University of Bahrain
College of Science
Mathematics department
First Semester 2009

Final Examination

Math 211
Duration: 2 hours
Date: 22 / 01 / 2009
Max. Mark: 50

Name:	
ID Number:	**Section:**

Instructions:

1) **Please check that this test has 6 questions and 8 pages.**
2) **Write your name, student number, and section in the above box.**

Marking Scheme

Questions	Max. Mark	Mark. Obtained
1	8	
2	6	
3	6	
4	10	
5	10	
6	10	
Total	50	

Good Luck

Question 1: **[8 marks]**

In each question, **only one** statement is **true**, circle the **right** statement.

(i) If A is 4×4 matrix, and nullity(A) = 2, then

a) The reduced row-echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

b) Column-vectors of A are linearly independent.

c) $AX = O$ has only the trivial solution.

d) Rank(A^T) = 3.

e) Nullity(A^T) = 2.

(ii) Let T_A be the linear transformation, multiplication by $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ -2 & -4 & 3 & 1 \end{bmatrix}$

a) dim Im(T_A) = 2.

b) dim Ker(T_A) = 1.

c) T_A (1,-1,1,-1) = (0,1,2).

d) (1,2,3,4) $\in$ Ker(T_A).

e) dim Im(T_A) + dim Ker(T_A) = 3

(iii) If the characteristic polynomial of a matrix A is $P_A = \lambda^2 (\lambda - 1)(\lambda + 1)^2 (\lambda - 3)$, then

a) A is invertible.

b) det(A) = -3.

c) A is of size 5×5.

d) $A X = O$ has infinitely many solutions.

e) The dimension of the eigenspace corresponding to $\lambda = 1$ is 2.

(vi) If V is a vector space with a basis $B = \{ v_1 , v_2 , v_3 , v_4 \}$, then

a) $\{ v_1 , v_2 , v_3 \}$ is linearly dependent.

b) $\{ v_1 , v_2 , v_3 \}$ spans V.

c) $v_1 \notin \text{Span}\{ v_2 , v_3 , v_4 \}$.

d) $\{ v_1 , v_2 , v_3 , v_1 + v_3 \}$ is a basis of V.

e) $(v_1 + v_2 + v_3)_B = (1 , 0 , 1 , 1)$.

<u>Question 2:</u> **[6 marks]**

Show that the following system has a unique solution, then solve it by inverting the coefficient matrix

$$\begin{aligned} x + 2y + z &= 1 \\ x + 2y + 2z &= 1 \\ x + 3y + az &= 2 \end{aligned}$$

Question 3: **[3 + 3 marks]**

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 4 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.

a) Does $C \in Span\{\, A \;,\; B\}$?

b) Let $T : \mathrm{M}_{22} \to \mathbb{R}$ be a linear transformation such that $T(A) = 1$ and $T(B) = 1$.

Show that $C \in \mathrm{Ker}(T)$.

<u>Question 4</u> [10 marks]

A square matrix A is said to be **orthogonal** if $AA^T = I$.

a) Show that every orthogonal matrix is invertible.

b) Show that $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an orthogonal matrix.

c) Prove that, if A and B are two orthogonal matrices, then AB is an orthogonal matrix.

d) Prove that, if $A = P\,D\,P^{-1}$, where P is an orthogonal matrix and D is a diagonal matrix, then $A^T = A$.

<u>Question 5</u> [10 marks]

Let $T : \mathrm{P}_2 \to \mathrm{P}_1$ be a linear transformation defined as

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1) + (a_1 + a_2) x$$

a) Show that T is a linear transformation.

b) Is $1 + x \in \mathrm{R}(T)$?

c) Is $q = 1 + 2x - 2x^2 \in \mathrm{Ker}(T)$?

d) Find a basis of $\mathrm{Ker}(T)$.

e) Find the rank and the nullity of T.

Question 6: **[2+5+5 marks]**

Consider the following matrix $A = \begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}$

a) Find the eigenvalues of A.

b) Find a basis for each eigenspace of A, and conclude that A is diagonalizable.

c) Find A^n for every positive integer n.