University of Bahrain
College of Science
Mathematics department
Second Semester 2008-2009

Final Examination

Math 311 — **Max. Marks: 50**
Date: 20 / 06 / 2009 — **Duration: 2 hours**

Name:
ID Number:

Instructions:

1) **Please check that this test has 6 questions and 7 pages.**

2) **Write your name, student number, and section in the above box.**

Question	Max. Marks	Marks obtained
1	9	
2	9	
3	9	
4	9	
5	7	
6	7	
Total	50	

Good Luck

<u>Question 1:</u> [9 marks]

a) Determine the order of the element (5,6,3) in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{30} \times \mathbb{Z}_{12}$.

Solution:

In $\mathbb{Z}_{15}$, $o(5) = \dfrac{15}{\gcd(5,15)} = 3$. In $\mathbb{Z}_{60}$, $o(6) = \dfrac{30}{\gcd(6,30)} = 5$. In $\mathbb{Z}_{12}$, $o(3) = \dfrac{12}{\gcd(3,12)} = 4$

Thus, in $\mathbb{Z}_{15} \times \mathbb{Z}_{60} \times \mathbb{Z}_{12}$, we have o(5,6,3) = lcm(3,5,4) = 60.

b) Let G be a group of order 17. Show that $G \cong \mathbb{Z}/17\mathbb{Z}$.

Solution:

As $|G| = 17$ is a prime number, then G is cyclic. If a is a generator of G, then the function f: Z $\to G$ defined by $f(n) = a^n$ is an epimomorphism. Moreover, we have $\mathrm{Ker}(f) = \{n \in \mathbb{Z} : f(n) = e\} = \{n \in \mathbb{Z} : a^n = e\} = \{n \in \mathbb{Z} : o(a) = 17 / n\} = 17\mathbb{Z}$. By the first isomorphic theorem, we get $\mathbb{Z}/\mathrm{Ker}(f) \cong G$, that is $\mathbb{Z}/17\mathbb{Z} \cong G$.

c) Let $f : G \to G'$ be homomorphism of groups. If $|G| = 7$ and $|G'| = 11$, prove that f is the trivial homomorphism defined by $f(x) = e'$ for every $x \in G$.

Solution:

As $|G| = 7$ is a prime number, then G is cyclic. If a is a generator of G, then

$G = < a >$ with $o(a) = 7$. Let $x \in G$.

- If $x = e$, then $f(e) = e'$.
- If $x \neq e$, then $o(f(x)) / |G'| = 11$. Also we know that $o(f(x)) / o(x) = 7$, so $o(f(x))$

divides gcd(7,11) = 1. Thus $o(f(x)) =1$, that is $f(x) = e'$.

Hence $f(x) = e'$ for every $x \in G$.

<u>Question 2</u> [9 marks]

d) Let G be a finite group and H a subgroup of G such that $3 < (G : H) < 9$. If $| H | = 5$, find the possible values of $| G |$.

Solution:

By Lagrange's theorem, we have $| G | = (G : H) \, | H | = 5 \, (G : H)$, As $3 < (G : H) < 9$, then $15 < | G | < 25$. Therefore, the possible values of $| G |$ is 20 or 25 or 30 or 35 or 40.

e) Let H and K be two subgroups of G such that $| H |$ is a prime number. Prove that $H \cap K = \{e\}$ or $H \subseteq K$.

Solution:

$H \cap K$ is a subgroup of H. By Lagrange's theorem, we have $| H \cap K |$ divides $| H |$. Since $| H |$ is a prime number, then $| H \cap K | = 1$ or $| H \cap K | = | H |$.

- If $| H \cap K | = 1$, then $H \cap K = \{e\}$.
- If $| H \cap K | = | H |$, then $H \cap K = H$, so $H \subseteq K$.

f) $H = \{ a + b\sqrt{3} : a , b \in \mathbb{Z} \}$ is a subgroup of $(\mathbb{R}, +)$. Is it cyclic?

Solution:

Suppose, by way of contradiction, that H is cyclic generated by an element $\delta = u + v\sqrt{3}$, where $a , b \in \mathbb{Z}$. As $1 = 1 + 0\sqrt{3} \in H$, then $1 = r\delta = r\,(u + v\sqrt{3})$ for a positive integer r. By comparison, we get $u = \frac{1}{r}$ and $v = 0$, so $\delta = \frac{1}{r}$ Also $\sqrt{3} \in H$, then $\sqrt{3} = 0 + 1\sqrt{3} \in H$, then $\sqrt{3} = s\delta = \frac{s}{r} \in \mathbb{Q}$, a contradiction.

<u>Question 3:</u> [9 marks]

Let $G = \{ \begin{bmatrix} 1-r & -r \\ r & 1+r \end{bmatrix} : r \in \mathbb{Z} \}$

a) Prove that G is a subgroup of the Linear group $\mathrm{GL}(2)$.

b) Show that $f : G \to \mathbb{Z}$ defined by $f(\begin{bmatrix} 1-r & -r \\ r & 1+r \end{bmatrix}) = r$ is an isomorphism.

c) Deduce that G is cyclic and find its generators.

Solution: Set $M(r) = \begin{bmatrix} 1-r & -r \\ r & 1+r \end{bmatrix}$.

a) It is clear that G is a subset of $\mathrm{GL}(2)$ since every element of G is invertible (indeed, $\det(M(r)) = 1$) and that G is non-empty since $I = M(0) \in G$. To show that $G \leq \mathrm{GL}(2)$, it suffices to verify the following easy operations:

$$\forall\, r , s \in \mathbb{Z}, \quad M(r) . M(s) = M(r + s) \quad \text{and} \quad M(r)^{-1} = M(-r).$$

b)

f is a homomorphism: Let $M(r)$, $M(s) \in G$, we have

$$f(M(r) . M(s)) = f(M(r + s) = r + s = f(M(r)) \; f(M(s)).$$

f is onto: For every $r \in \mathbb{Z}$, we have $r = f(M(r))$, where $M(r) \in G$.

f is one-to-one:

$$\begin{aligned} \mathrm{Ker}(f) &= \{ M(r) \in G : \; f(M(r)) = 0 \} \\ &= \{ M(r) \in G : \; r = 0 \} \\ &= \{ M(0)\} = \{ I \} \end{aligned}$$

c) We have $G \cong \mathbb{Z}$. As $\mathbb{Z}$ is cyclic and has two generators 1 and -1, then G is cyclic and has two generators : $M(1) = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ and $M(-1) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

Question 4: **[9 marks]**

Let f be the homomorphism $f\colon \mathbb{Z} \to S_4$ of groups such that $f(1) = (1\ 3)(3\ 4)$.

a) Give the definition of f.

b) Find $K = Ker(f)$ and $\mathbb{Z} / K$.

c) Find $\mathrm{Im}(f)$ and the bijective correspondence between $\mathbb{Z} / K$ and $\mathrm{Im}(f)$.

Solution:

a) Let $\sigma = (1\ 3)(3\ 4) = (1\ 3\ 4)$. For $n \in \mathbb{Z}$, we have $f(n) = [f(1)]^n = \sigma^n$.

b) First note that $o(\sigma) = 3$. Then

$$
\begin{aligned}
K = \mathrm{Ker}(f) &= \{ n \in \mathbb{Z} : f(n) = e \} \\
&= \{ n \in \mathbb{Z} : \sigma^n = e \} \\
&= \{ n \in \mathbb{Z} : o(\sigma) \ / \ n \} \\
&= \{ n \in \mathbb{Z} : 3 \ / \ n \}
\end{aligned}
$$

Thus $K = n\mathbb{Z}$ and $\mathbb{Z} / K = \mathbb{Z} / n\mathbb{Z}$.

c) We have $f(0) = e$, $f(1) = \sigma = (1\ 3)(3\ 4)$ and $f(2) = \sigma^2 = (1\ 4\ 3)$. Then

$$
\begin{aligned}
\mathrm{Im}(f) &= \{ f(0) , f(1) , f(2) \} \\
&= \{ e , \sigma = (1\ 3\ 4) , \sigma^2 = (1\ 4\ 3) \}
\end{aligned}
$$

d) The function $\bar{f} : \mathbb{Z} / n\mathbb{Z} : \to \mathrm{Im}(f)$ defined by

$$\bar{f}(\bar{0}) = e, \ \bar{f}(\bar{1}) = (134) \text{ and } \bar{f}(\bar{2}) = (143) .$$

is the bijective correspondence between $\mathbb{Z} / K$ and $\mathrm{Im}(f)$.

Question 5: **[3 + 2 + 2 marks]**

Let G be an Abelian group and $H = \{a \in G : o(a) \text{ is finite }\}$.

a) Prove that H is a normal subgroup of G.

b) Prove that if $a \in H$, then $o(aH)$ divides $o(a)$.

c) Deduce $o(aH)$ is finite if and only if $a \in H$.

Solution:

a) Since G is an Abelian group, it suffices to show that H is a subgroup of G. It is clear that $o(e) = 1$, so $e \in H$ and H is non-empty.

Let $x , y \in H$. Then $o(x) = r < \infty$ and $o(y) = s < \infty$. We have

(i) $(xy)^{r\,s} = [(x)^r]^s\,[(y)^s]^r = (e)^s\,(e)^r = e$, so $o(xy) / r\,s$. Thus $o(xy) < \infty$, and $xy \in H$.

(ii) $(x^{-1})^r = [(x)^r]^{-1} = (e^{-1})^r = e$, so $o(x^{-1}) / r$. Thus $o(x^{-1}) < \infty$, and $x^{-1} \in H$.

b) Let $o(a) = r$. We have $(aH)^r = a^r H = eH = H$, so $o(aH)$ divides $o(a)$.

c) In light of the point (b), we can say that, if $o(a)$ is finite, then $o(aH)$ is finite. Conversely, suppose that $o(aH) = s$ is finite. Then $(aH)^s = a^s H = H$, so $a^s \in H$. It follows, by definition of elements H, that $o(a^s) = t$ is finite. Thus $(a^s)^t = a^{st} = e$. Hence $o(a)$ divides st, and $a \in H$.

<u>Question 6:</u> **[4 + 3 marks]**

Let G be a finite group of order $(15)^2$.

a) Determine the number of Sylow 3-subgroups and the number Sylow 5-subgroups of G.

b) Is G a simple group?

Solution:

a) We have $| G | = (3)^2(5)^2$. Let N_3 the number of Sylow 3-subgroups and N_5 the number of Sylow 5-subgroups.

By application of the third Sylow's Theorem, N_3 and N_5 are divisors of $| G |$, so the possible of their values are

$$1,\ 3,\ 3^2,\ 5,\ (3)(5),\ (3)^2 5,\ 5^2,\ 3(5)^2 \text{ and } (3)^2(5)^2.$$

- As $N_3 \equiv 1 \pmod 3$, then $N_3 = 1$ or $N_3 = 25$.

- As $N_5 \equiv 1 \pmod 5$, then $N_5 = 1$.

b) According to (a), G has a unique Sylow 5-subgroup P. By application of the second Sylow's Theorem, we can say that $g^{-1}Pg$ is also a Sylow 5-subgroup for every $g \in G$. Thus, $g^{-1} Pg = P$, and P is a normal subgroup of G of order 25. Hence, G is not simple.