University of Bahrain
College of Science
Mathematics department
First Semester 2008-2009

Final Examination

Math 312
Duration: 2 hours
Date: 22 / 01 / 2009
Max. Mark: 50

Name:	
ID Number:	**Section:**

Instructions:

1) **Please check that this test has 6 questions and 8 pages.**
2) **Write your name, student number, and section in the above box.**

Marking Scheme

Questions	Max. Mark	Mark. Obtained
1	8	
2	8	
3	8	
4	8	
5	8	
6	10	
Total	50	

Good Luck

<u>Question 1:</u> **[4 + 4 marks]**

a) Let R be an integral domain with identity. Prove that if p is irreducible and u is a unit, then pu is irreducible.

b) Let $f(x) = x^4 + n\,x^3 + x^2 + n \in \mathbb{Z}_7[\mathrm{x}]$. Find n if the polynomial $g(x) = x - 2$ divides $f(x)$.

Question 2: **[4 + 4 marks]**

a) Find all irreducible polynomials of degree 2 in the polynomial rings $\mathbb{Z}_2[x]$.

b) Show that, if $a + bi$ is prime in $\mathbb{Z}[i]$, then $a - bi$ is prime in $\mathbb{Z}[i]$.

Question 3: **[4 + 4 marks]**

a) Let R be an Euclidean domain with degree function δ. Prove that if $\delta(1) \geq -2$, then the function $\delta': R - \{0\} \to \mathbb{N}$ defined by $\delta'(x) = \delta(x) + 2$ is also a degree function.

b) Let $P = X^2 + 2X + 2$ be a polynomial of $\mathbb{Z}_3[X]$. Show that $\mathbb{Z}_3[X]/(P)$ is a field and find its elements.

<u>Question 4</u> [4 + 4 marks]

a) Let R be division ring. Prove that the center $Z(R)$ of R is a field, where $Z(R)$ is defined by $Z(R) = \{a \in R : a\,x = x\,a \text{ for all } x \in R\}$.

b) Let R be a Boolean ring and P be a prime ideal of R. Show that R/P has only two elements (use the fact that $x^2 = x$ for all $x \in R$). Then conclude that P is a maximal ideal.

<u>Question 5</u> [4 + 4 marks]

Let R be a principal ideal domain and P be a prime ideal of R.

a) Prove that P is generated by a prime element.

b) Prove that P is a maximal ideal.

<u>Question 6</u> [3 + 3 + 4 marks]

Let R be the set of all upper triangular matrices $R = \{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} : x , y \in \mathbb{Z} \}$.

a) Prove that R is a ring with identity.

b) Prove that $J = \{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} : a \in \mathbb{Z} \}$ is an ideal of R.

c) By considering the function $f : R \to \mathbb{Z}\times\mathbb{Z}$ defined by $f(\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}) = (x , z)$, show that J is not a prime ideal of R.