University of Bahrain
College of Science
Mathematics department
First Semester 2009-2010

Final Examination

Math 312
Duration: 2 hours
Date: 19 / 01 / 2010
Max. Mark: 50

Name:	
ID Number:	**Section:**

Instructions:

1) **Please check that this test has 3 questions and 9 pages.**
2) **Write your name, student number, and section in the above box.**

Marking Scheme

Questions	Max. Mark	Mark. Obtained
Q1	13	
Q2- Part I	7	
Q2- Part II	8	
Q3- Part I	10	
Q3- Part II	12	
Total	50	

Good Luck

Question 1: **[2+3+2+3+3 marks]**

Let $Q = X^2 + X + 1$ be a polynomial of $\mathbf{Z}_2[X]$.

a) Show that $K = \mathbf{Z}_2[X]/(Q)$ is a field.

b) Let $u = \overline{X}$ be the equivalence class of X modulo (Q). Find the elements of K and give the addition and multiplication tables of K.

c) Simplify $(1 + u)^3$ in K.

d) Find a prime number p so that the polynomial $f(X) = X^4 + X^3 + 3X + (p - 2)$ is divisible by $X - 2$ in $\mathbf{Z}_p[X]$.

e) Find GCD$(X^4 + 2X^3 + 2 , 2X^2 + X + 1)$ in $\mathbf{Z}_5[X]$.

<u>Question 2:</u>

Part I: [2+2+3 marks]

Let R be the integral domain $R = \mathbf{Z}[\sqrt{-2}\,]$ and $N : R \to \mathbf{N}$ the function defined by $N(a + b\sqrt{-2}\,) = a^2 + 2b^2$.

a) Find the units of R.

b) Prove that if $N(\alpha)$ is a prime number, then α is irreducible.

c) Show that $\alpha = 5$ is irreducible, but $N(\alpha)$ is not a prime number ?

Part II: [2+4+2 marks]

d) Prove that $R = \mathbf{Z}[\sqrt{-2}\,]$ is an Euclidean domain with $\delta(\alpha) = N(\alpha)$ for every $\alpha \neq 0$.

e) Let α and β be two nonzero elements of R. Prove that if α divides β and $N(\alpha) = N(\beta)$, then $\alpha = \pm\beta$.

f) Let $\gamma = 1 + 3\sqrt{-2}$. Prove that the ideal (γ) is maximal in R.

Question 3:

Part I: [5+2+3 marks]

Let R be a commutative ring with identity 1 and J a nonzero ideal of R. Consider the cartesian product $S = R \times J$. Define on S the following binary operations:

$$(r\ ,\ a) + (s\ ,\ b) \ = \ (r + s\ ,\ a + b)$$

$$(r\ ,\ a)\ .\ (s\ ,\ b) \ = \ (r s\ ,\ r b + s a)$$

a) Prove that S is a commutative ring with identity.

b) Is S an integral domain ?

c) Let H be a subring of R. Prove that $H \times J$ is a subring of S

Part II: [3+2+3+2+2=12 marks]

For an ideal I of R, we define $T(I) = I \times J$.

d) Prove that $T(I)$ is an ideal of S.

e) Prove that the function $f : S \to R / I$ defined by $f(r , a) = \bar{r}$ is a homomorphism.

f) Show that $S / T(I) \cong R/I$.

g) Deduce that I is a maximal ideal of R if and only if $T(I)$ is a maximal ideal of S.

h) Is $4\mathbf{Z} \times \mathbf{Z}$ a maximal ideal of $\mathbf{Z} \times \mathbf{Z}$?