**University of Bahrain**
**College of Science**
**Mathematics department**
**Summer Semester 2008**

## Final Examination (2)

**Math 211**
**Duration: 2 hours**
**Date: 25 / 06 / 2007**
**Max. Mark: 50**

| | |
|---|---|
| **Name:** | |
| **ID Number:** | **Section:** |

**Instructions:**

1) **Please check that this test has 7 questions and 9 pages.**
2) **Write your name, student number, and section in the above box.**

### Marking Scheme

| Questions | Max. Mark | Mark. Obtained |
|---|---|---|
| 1 | 9 | |
| 2 | 5 | |
| 3 | 6 | |
| 4 | 11 | |
| 5 | 6 | |
| 6 | 9 | |
| 7 | 4 | |
| Total | 50 | |

**Good Luck**

**Question 1:** **[ 3+3+3 marks]**

Consider the following matrix $A = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix}$

**a)** Find the eigenvalues of $A$.

**b)** Find a basis for each eigenspace of $A$.

**c)** Find $A^n$ for every positive integer $n$.

**<u>Question 2:</u>** **[ 5 marks]**

Solve the following system by inverting the coefficient matrix

$$\begin{aligned} x + 2y + z &= 3 \\ x - y + z &= 1 \\ x + y &= 2 \end{aligned}$$

**<u>Question 3</u> [ 3+3 marks]**

**a)** Find a **diagonal** matrix $X$ such that $AX^2 + BX + C = O$, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, \quad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

**b)** Show that every vector $(x , y , z) \in \text{Span}\{ (-1 , 1 , -1) , (-1 , 3 , 2) \}$ solves the equation $5x + 3y - 2z = 0$.

**<u>Question 4</u>** **[2+3+3+3 marks]**

Let $T : \mathrm{P}_2 \to \mathrm{P}_3$ be a linear transformation defined as

$T(a_0 + a_1 x + a_2 x^2) = (a_1 + a_2) x + (2 a_1 + 2 a_2) x^2 + (3 a_1 + 3 a_2) x^3$

**a)** Is $x + 2x^2 + 4x^3 \in \mathrm{R}(T)$ ?

**b)** Find a basis of Ker($T$).

**c)** Find a basis of R($T$).

**d)** If $B = \{ 1 , x , x^2 \}$ is a basis of $\mathrm{P}_2$ and $B' = \{ 1 , x , x^2 , x^3 \}$ is a basis of $\mathrm{P}_3$, find the matrix of $T$ with respect to the bases $B$ and $B'$.

**<u>Question 5:</u> [ 3 + 3 marks]**

**a)** Let $B$ be a basis of $P_1$. If $(2 - x)_B = (1, 2)$ and $(4 - 5x)_B = (-1, 1)$, find the basis $B$.

**b)** Let $T : \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation with $T[(2, 3, 1)] = 4$ and $T[(4, 1, 3)] = 7$.

Find $T[(2, -7, 3)]$ ( **Hint**: $(2, -7, 3) \in \text{Span}\{(2, 3, 1), (4, 1, 3)\}$ )

**<u>Question 6:</u> [ 6× 1.5 marks]**

Complete the following statements:

**(i)** Let $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & k \end{bmatrix}$. If $A$ satisfies $AA^T = I$, then $k =$ ____________

**(ii)** If $V$ is the subspace of $M_{22}$ that consists of all matrices of the form $\begin{bmatrix} a & a+b \\ a-2b & d \end{bmatrix}$, then dim($V$) = _____________

**(iii)** If **-1** and **3** are the eigenvalues of a matrix $A$, then the eigenvalues of $A^4$ are ___________

**(iv)** If $A$ is a $4 \times 6$ matrix, then the smallest possible value for the nullity of $A$ is ___________

**(v)** If $A = A^3$, then the possible eigenvalues of $A$ are __________________

**(vi)** If the matrix $\begin{bmatrix} 1 & a \\ 1 & 5 \end{bmatrix}$ has only **one** eigenvalue, then $a =$ _____________

**<u>Question 7:</u> [ 4 marks]**

In each question, **only one** statement is false, circle the **false** statement.

**(i)** If the characteristic polynomial of a matrix $A$ is $P_A = (\lambda - 1)(\lambda + 2)^2(\lambda - 5)$, then

a) $A$ is invertible.

b) $\det(A) = -20$.

c) $A$ is of size $4 \times 4$.

d) $A X = O$ has infinite solutions.

e) $A$ is diagonalizable only if the dimension of the eigenspace corresponding to $\lambda = -2$ is 2

**(ii)** If the dimension of a vector space $V$ is $n$, then

a) Any set of more than $n$ vectors in $V$ is linearly dependent.

b) Any set of less than $n$ vectors in $V$ does not span $V$.

c) Any linearly independent set of $n$ vectors in $V$ is a basis of $V$.

d) Any set of $n$ vectors in $V$ that contains the zero vector can not span $V$.

e) Only if $B$ is a basis, any vector $u$ in $V$ can be written as a linear combination of a set of vectors of $B$.

**(iii)** Given the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

a) The eigenvalues of $A$ are 1 , -1 , 2.

b) $A$ is diagonalizable.

c) $\det(A) = -2$.

d) $\operatorname{rank}(A) < 3$.

e) The sum of dimensions of all eigenspaces of $A$ is 3.

**(vi)** If $A$ is $4 \times 5$ matrix, and rank($A$) = 4, then

a) Row-vectors of $A$ are linearly independent.

b) Column-vectors of $A$ are linearly independent.

c) $AX = O$ has infinite solutions.

d) Nullity($A$) = 1.

e) Nullity($A^T$) = 0.