**University of Bahrain**
**College of Science**
**Mathematics department**
**Second Semester 2003-2004**

**<u>Final Examination</u>**

**Math 312** — **Max. Mark: 50**
**Duration: 2 hours** — **Date: 19 th June, 2004**

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**<u>Question 1:</u> [ 19 marks]**

Let $R$ be a commutative ring with identity 1 and $J$ a nonzero ideal of $R$. Consider the cartesian product $S = R \times J$. Define on $S$ the following binary operations:

$$(r , a) + (s , b) = (r + s , a + b)$$

$$(r , a) \,.\, (s , b) = (r s , r b + s a)$$

**a)** Prove that $S$ is a commutative ring with identity.

**b)** Is $S$ an integral domain ?

For an ideal $I$ of $R$, we define $T(I) = I \times J$.

**c)** Prove that $T(I)$ is an ideal of $S$.

**d)** Prove that the function $f : S \to R / I$ defined by $f(r , a) = \bar{r}$ is a homomorphism.

**e)** Show that $S / T(I) \cong R/I$ .

**f)** Deduce that $I$ is a maximal ideal of $R$ if and only if $T(I)$ is a maximal ideal of $S$.

**g)** Is $5\mathbb{Z} \times \mathbb{Z}$ a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$?

**<u>Question 2:</u> [ 12 marks]**

In the polynomial ring $\mathbb{Z}_3[X]$, consider the polynomial $Q = X^2 + 2X + 2$.

**a)** Prove that $Q$ is an irreducible polynomial of $\mathbb{Z}_3[X]$.

**b)** Show that $K = \mathbb{Z}_3[X]/(Q)$ is a field.

**c)** Set $u = \overline{X}$ the equivalence class of $X$ modulo $(Q)$, find all elements of $K$.

**d)** Simplify $(1 + 2u)^3$ in $K$.

**<u>Question 3:</u> [ 12 marks]**

Let $R$ be a commutative ring with identity 1, $I$ be an ideal of $R$. and $f : R \to R$ an onto homomorphism. Define $\sqrt{I} = \{ x \in R : x^n \in I$ for some positive integer $n \}$

**a)** Prove that $\sqrt{I}$ is an ideal of $R$.

**b)** Prove that $\sqrt{\sqrt{I}} = \sqrt{I}$ .

**c)** Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$ for every two ideals $I$ and $J$ of $R$.

**d)** Prove that if, Ker($f$) $\subseteq I$, then $f(\sqrt{I}) = \sqrt{f(I)}$ .

**<u>Question 4:</u> [7 marks]**

Prove that $R$ = Z[ $\sqrt{2}$ ] is a Euclidean domain

( Consider the function δ: $R$-{0} → IN defined by $\delta(a + b\sqrt{2}) = | a^2 - 2b^2 |$ )