University of Bahrain
College of Science
Mathematics department
First Semester 2007-2008

Final Examination

Math 312
Duration: 2 hours
Date: 23 / 01 / 2008
Max. Mark: 50

Name:	
ID Number:	**Section:**

Instructions:

1) **Please check that this test has 5 questions and 7 pages.**
2) **Write your name, student number, and section in the above box.**

Marking Scheme

Questions	Max. Mark	Mark. Obtained
1	8	
2	6	
3	12	
4	12	
5	12	
Total	50	

Good Luck

Question 1: **[3+3+2 marks]**

a) Let x be a nonzero element of a ring R with identity such that $x^3 = 0$. Show that $u = 1 - x$ is a unit of R, and find the principal ideal of R generated by u.

b) Let R be an Euclidean domain with degree function δ. Show that the function $\delta' : R - \{0\} \to \mathbf{N}$, defined by $\delta'(x) = a\,\delta(x)$ is also a degree function for every $a > 0$.

c) Find the set of units of the polynomial ring $R[x]$, where $R = \mathbf{Z}[\sqrt{-5}]$.

<u>Question 2:</u> [3+3 marks]

a) Let p be a prime number. Show that p is not irreducible in $\mathbf{Z}[i]$ if and only if p can be written as $p = a^2 + b^2$ for some integers a and b.

b) Let M and N be two distinct maximal ideals of a ring R. Show that $M + N = R$.

Question 3: **[4+4+4 marks]**

Let $f(x) = x^2 + a\,x + b \in \mathbf{Z}_2[x]$**.**

a) Prove that $f(x)$ is irreducible if and only if $a = b = 1$.

b) Suppose that $a = b = 1$. Prove that $F = \mathbf{Z}_2[x]/(\,f(x))$ is a field and finds its elements.

c) Give the addition and multiplication tables of F.

Question 4: **[3+3+3+3 marks]**

Let $\varphi : \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z} / n\mathbf{Z}$ be the function defined by $\varphi(x, y) = \bar{x}$.

a) Prove that φ is a homomorphism.

b) Prove that $\text{Ker}(\varphi) = n\mathbf{Z} \times \mathbf{Z}$.

c) Show that $(\mathbf{Z} \times \mathbf{Z}) / (n\mathbf{Z} \times \mathbf{Z}) \cong \mathbf{Z} / n\mathbf{Z}$.

d) Show that $n\mathbf{Z} \times \mathbf{Z}$ is a maximal ideal of $\mathbf{Z} \times \mathbf{Z}$ if and only if n is a prime number.

<u>Question 5</u>: **[3+3+3+3 marks]**

Let $\boldsymbol{I} = \{p(x) \in \mathbf{Z}[x] : p(0) \text{ is even } \}$.

a) Show that $\boldsymbol{I}$ is an ideal of $\mathbf{Z}[x]$.

b) Let $f(\mathrm{x}) = 2x^3 + 3x + 6$. Show that $f(x)$ is irreducible in $\mathbf{Z}[x]$, and that $f(x) \in \boldsymbol{I}$.

c) Show that, if $\boldsymbol{I}$ is a principal ideal, then $\boldsymbol{I}$ is generated by $f(x)$.

d) Conclude that $\boldsymbol{I}$ can not be principal, and R is not a PID.